manifold. The feedback gains and synchronous protocol gain depend on the output regulation problem. The permissible region of the eigenvalue distribution to ensure the stability of synchronous manifold is a transversal band along the right real axis. A numerical example illustrates the efficacy of the presented theoretical analysis.

A natural extension of this work will be the error feedback case which, in the classic output regulation problem, is solvable if the full information case is solvable. The difficulty is how to rationally formulate the error feedback case in the distributed sense. One possible formulation is that leader nodes have the error information $e = Cx + Qw$ while follower nodes the sum of weighted output error with respect to its neighboring nodes $e = \sum a_{ij}(Cx_i - Cx_j)$. However, the trivial extension of the framework developed in this note is infeasible for such a formulation. Whether there exists a framework in which the similar results as in the classic output regulation problem still hold is an interesting future work.

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Activity Invariant Sets and Exponentially Stable Attractors of Linear Threshold Discrete-Time Recurrent Neural Networks

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***Abstract*—This technical note proposes to study the activity invariant sets and exponentially stable attractors of linear threshold discrete-time recurrent neural networks. The concept of activity invariant sets deeply describes the property of an invariant set by that the activity of some neurons keeps invariant all the time. Conditions are obtained for locating activity invariant sets. Under some conditions, it shows that an activity invariant set can have one equilibrium point which attracts exponentially all trajectories starting in the set. Since the attractors are located in activity invariant sets, each attractor has binary pattern and also carries analog information. Such results can provide new perspective to apply attractor networks for applications such as group winner-take-all, associative memory, etc.**

***Index Terms*—Activity invariant sets, discrete-time recurrent neural networks, exponentially stable attractors, linear threshold.**

I. INTRODUCTION

In recent years, linear threshold recurrent neural networks (LT networks) have been studied by many authors [7], [9], and [16]. The linear threshold transfer function is an unbounded function with binary pattern. It has been used to model many cortical neural networks [1]–[4]. Networks endowed with this transfer function form a class of hybrid analog and digital networks that can implement a form of hybrid analog-digital computation. Since the linear threshold transfer function is essentially nonlinear, complex dynamic properties may exist in such networks [12] and [17]–[19]. LT networks have been got many applications, such as associative memory [10], [11], winner-take-all [5], group selection [6], [14], feature binding [13], etc.

The main contributions of this technical note consist of two parts. We fist present the concept of activity invariant set for discrete-time LT networks. Discrete-time recurrent neural networks can provide direct algorithms and easily be implemented by digital hardware [15]. Moreover, invariant sets play important roles in dynamics study of recurrent neural networks. An invariant set restricts trajectories starting from the set stay in the set. The concept of activity invariant set more deeply describes the dynamic properties of invariant sets: the activity of some neurons keeps invariant during the time evolution. Thus, neurons can be divided into two classes by active neurons and inactive neurons. We will derive conditions for locating activity invariant sets.

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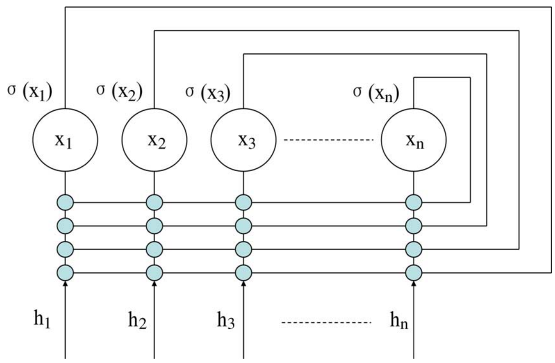

Fig. 1. Architecture of recurrent network (1).

In applying of recurrent neural networks to the application of associative memory, it is crucial that the networks have stable attractors. Stable attractors stored as memories to the networks are often used to implement associative memory [8]. The memories can be recalled by encoding initial conditions as computational inputs to the network. Thus, based on the concept of activity invariant set, we will show that under some conditions, an activity invariant set has one equilibrium point which attracts exponentially all the trajectories in the invariant set, i.e., it has an exponentially stable attractor. Such attractors are located in activity invariant sets, thus each attractor has binary pattern and also carries analog information. This is quite interesting since these attractors could be used to store memories with both binary and analog information. It is believed that these results can have potential applications such as group winner-take-all, associative memory, etc. In the application of group winner-take-all, the network outputs are required to have binary pattern, i.e., the winners and the losers. In addition, there may exist differences among neurons in the winner group, such differences can be depicted by analogy information of each neuron in the winner group.

The rest of this technical note is organized as follows. In Section II, we present some preliminaries. Main results about activity invariant sets and exponentially stable attractors are given in Section III. Simulations are carried out in Section IV to illustrate the theory. Conclusions are given in Section V.

II. Preliminaries

In this technical note, we study a class of discrete recurrent neural network with unsaturating linear threshold transfer functions described by the following nonlinear discrete equations:

$$x_i(k+1) = \sum_{j=1}^{n} a_{ij}\sigma\left(x_j(k)\right) + h_i, \quad (i = 1, 2, \cdots, n) \qquad (1)$$

for $k \geq 0$, where each x_i denotes the activity of neuron i, and $x = (x_1, \cdots, x_n)^T \in R^n$. $\sigma(\cdot)$ is the unsaturating linear threshold activation function defined by

$$\sigma(s) = \max\{0, s\}, \quad s \in R$$

and $\sigma(x_i)$ denotes the output of neuron i, $\sigma(x) = (\sigma(x_1), \sigma(x_2), \cdots, \sigma(x_n))^T$, $a_{ij}(i, j = 1, 2, \cdots, n)$ are connection weights which are constants, $h_i(i = 1, 2, \cdots, n)$ denotes external input.

Fig. 1 shows the architecture of network (1) which is a kind of recurrent neural networks. In network (1), each neuron connects with all the neurons, that is, the input of each neuron i is composed of the external input h_i and the outputs of all neurons (including itself) weighing of the connection weights $a_{ij}(i, j = 1, 2, \cdots, n)$.

The vector form of (1) can be written as

$$x(k+1) = W\sigma\left(x(k)\right) + h$$

for $k \geq 0$. Given any $x(0) \in R^n$, we denote by $x(k, x(0))$ the trajectory of (1) starting from $x(0)$.

Definition 1: A neuron with index i is active at time k, if $x_i(k) > 0$. A neuron with index i is inactive at time k, if $\sigma(x_i(k)) = 0$.

Definition 2: A set $D \subset R^n$ is called an invariant set of (1), if each trajectory starting in D will remain in D for ever.

Definition 3: A set $D \subset R^n$ is called an activity invariant set of (1), if D is an invariant set, and given any $x(0) \in D$ it holds that

$$\begin{cases} x_i(k) > 0, & \text{if } x_i(0) > 0 \\ \sigma\left(x_i(k)\right) = 0, & \text{if } \sigma\left(x_i(0)\right) = 0 \end{cases}$$

for all $k \geq 0$.

Activity invariant set says that in an invariant set, the activity of a neuron keeps invariant, i.e., if a neuron is initially active then it keeps active for all the time $k \geq 0$, if a neuron is initially inactive then it keeps inactive thereafter.

A point $x^* \in R^n$ is called an equilibrium point of (1) if

$$x_i^* = \sum_{j=1}^{n} a_{ij}\sigma\left(x_j^*\right) + h_i, \quad (i = 1, \cdots, n).$$

Given any $x \in R^n$, denote $\|x\| = \max_{1 \leq i \leq n}\{|x_i|\}$.

Definition 4: An invariant set D of (1) is said to have an exponentially stable attractor x^*, if x^* is an equilibrium point of (1), and there exit constants $M > 0$ and $\epsilon > 0$ such that for any $x(0) \in D$, it holds that

$$\|x\left(k, x(0)\right) - x^*\| \leq M \cdot \|x(0) - x^*\| \cdot e^{-k\epsilon}$$

for all $k \geq 0$.

An invariant set D has exponentially stable attractor x^* implies that any trajectory starting in D will converge exponentially to the equilibrium point x^*.

Lemma 1: If an invariant set D has an exponentially stable attractor x^*, then D cannot have another exponentially stable attractor different from x^*.

Proof: By Definition 4, for any $x(0) \in D$, it holds that

$$\|x\left(k, x(0)\right) - x^*\| \leq M \cdot \|x(0) - x^*\| \cdot e^{-k\epsilon}$$

for all $k \geq 0$.

Suppose the invariant set D has another exponentially stable attractor $x^\dagger$. Then, the trajectory starting from $x^\dagger$ satisfies that $x^\dagger = x(k, x^\dagger)$ for all $k \geq 0$. Thus

$$\left\|x^\dagger - x^*\right\| \leq M \cdot \|x^\dagger - x^*\| \cdot e^{-k\epsilon}$$

for all $k \geq 0$. Clearly, $x^\dagger = x^*$. The proof is complete. ■

Lemma 2: Suppose that D is an invariant set of (1). If there exist two constants $M > 0$ and $\epsilon > 0$ such that for any $\bar{x}(0) \in D, \hat{x}(0) \in D$, it holds that

$$\|x\left(k, \bar{x}(0)\right) - x\left(k, \hat{x}(0)\right)\| \leq M \cdot \|\bar{x}(0) - \hat{x}(0)\| \cdot e^{-k\epsilon} \qquad (2)$$

for all $k \geq 0$, then the invariant set D has an exponentially stable attractor.

Proof: Choosing a constant $m > 0$ such that

$$Me^{-m\epsilon} < \frac{1}{2}. \qquad (3)$$

Given any $\eta > 0$, we can select a constant $K > 0$ such that if $m \geq K$, it holds that

$$4Me^{-m\epsilon} \|x(m, \bar{x}(0)) - \bar{x}(0)\| < \eta. \quad (4)$$

Given any constant $p > 0$, from (2) and (3), it can be calculated that

$$\begin{aligned}
&\|x(k+p, \bar{x}(0)) - x(k, \bar{x}(0))\| \\
&\quad \leq \|x(k+p, \bar{x}(0)) - x(k+p+m, \bar{x}(0))\| \\
&\qquad + \|x(k+p+m, \bar{x}(0)) - x(k+m, \bar{x}(0))\| \\
&\qquad + \|x(k+m, \bar{x}(0)) - x(k, \bar{x}(0))\| \\
&\quad = \|x(k+p, \bar{x}(0)) - x(k+p, x(m, \bar{x}(0)))\| \\
&\qquad + \|x(m, x(k+p, \bar{x}(0))) - x(m, x(k, \bar{x}(0)))\| \\
&\qquad + \|x(k, x(m, \bar{x}(0))) - x(k, \bar{x}(0))\| \\
&\quad \leq Me^{-(k+p)\epsilon} \|\bar{x}(0) - x(m, \bar{x}(0))\| \\
&\qquad + Me^{-m\epsilon} \|x(k+p, \bar{x}(0)) - x(k, \bar{x}(0))\| \\
&\qquad + Me^{-k\epsilon} \|x(m, \bar{x}(0)) - \bar{x}(0)\| \\
&\quad \leq 2Me^{-k\epsilon} \|x(m, \bar{x}(0)) - \bar{x}(0)\| \\
&\qquad + \frac{1}{2} \|x(k+p, \bar{x}(0)) - x(k, \bar{x}(0))\|
\end{aligned}$$

for all $k \geq 0$. Then, using (4), it follows that

$$\begin{aligned}
\|x(k+p, \bar{x}(0)) - x(k, \bar{x}(0))\| &\leq 4Me^{-k\epsilon} \|x(m, \bar{x}(0)) - \bar{x}(0)\| \\
&\leq \eta
\end{aligned}$$

for all $k \geq K$. By the well known *Cauchy Convergence Principle*, there must exist a x^* such that

$$\lim_{k \to +\infty} x(k, \bar{x}(0)) = x^*.$$

Clearly, $x^* \in D$ is an equilibrium point of the network (1) in the region D, thus, $x(k, x^*) = x^*$ for all $k \geq 0$. Then, it holds that

$$\|x(k, \hat{x}(0)) - x^*\| \leq M \cdot \|\hat{x}(0) - x^*\| \cdot e^{-k\epsilon}$$

for all $k \geq 0$.

By Definition 4, the invariant set D has an exponentially stable attractor x^*. Using Lemma 1, the network (1) cannot have exponentially stable attractor different from x^*. The proof is complete. ■

Given a constant c, denote by

$$\begin{cases} c^- = \min\{c, 0\} \\ c^+ = \max\{c, 0\}. \end{cases}$$

Clearly, $c^- \leq 0, c^+ \geq 0$.

Lemma 3: It holds that

$$c^+ - c^- = |c|, \quad c^+ \times c^- = 0.$$

■

Proof: The proof is trivial.

III. Activity Invariant Sets and Exponentially Stable Attractor

In this section, we are going to establish conditions to locate activity invariant sets. We will address the problems: under what conditions the network (1) can have invariant sets? Can an invariant set have an exponentially stable attractor?

Theorem 1: Suppose that $P \cup N = \{1, 2, \cdots, n\}$ and $P \cap N$ is empty. If there exist constants $0 < \xi_i < \eta_i (i \in P)$ such that

$$\begin{cases} \sum_{j \in P} \left(a_{ij}^+ \xi_j + a_{ij}^- \eta_j\right) + h_i > \xi_i, \\ \sum_{j \in P} \left(a_{ij}^+ \eta_j + a_{ij}^- \xi_j\right) + h_i < \eta_i, \end{cases} \quad (i \in P) \quad (5)$$

and

$$\sum_{j \in P} \left(a_{lj}^+ \eta_j + a_{lj}^- \xi_j\right) + h_l < 0, \quad (l \in N) \quad (6)$$

then the set

$$D = \{x | x_i \in (\xi_i, \eta_i), i \in P; \quad \sigma(x_l) = 0, l \in N\}$$

is an activity invariant set of the network (1), and the neurons with index in P are active invariant, the neurons with index in N are inactive invariant. Moreover, D has an exponentially stable attractor.

Proof: The proof will be divided into two parts. In the first part, we will prove that D is an invariant set, i.e., given any initial $x(0) \in D$, the trajectory $x(k) (k \geq 0)$ starting from $x(0)$ has the property for all $k \geq 0$ that $\xi_i < x_i(k) < \eta_i$ if $i \in P$, and $x_l(k) < 0$ if $l \in N$.

We will show this by mathematical induction. Since $x(0) \in D$, suppose it was also true that $x(k) \in D (k \geq 0)$, we will show that $x(k+1) \in D (k \geq 0)$ correspondingly.

It follows from (1) and by condition (5) that

$$\begin{aligned}
x_i(k+1) &= \sum_{j \in P} a_{ij} x_j(k) + h_i \\
&\geq \sum_{j \in P} \left(a_{ij}^+ \xi_j + a_{ij}^- \eta_j\right) + h_i \\
&> \xi_i
\end{aligned}$$

and

$$\begin{aligned}
x_i(k+1) &= \sum_{j \in P} a_{ij} x_j(k) + h_i \\
&\leq \sum_{j \in P} \left(a_{ij}^+ \eta_j + a_{ij}^- \xi_j\right) + h_i \\
&< \eta_i
\end{aligned}$$

for all $i \in P$ and $k \geq 0$. On the other hand, from (1) and condition (6), it gives that

$$\begin{aligned}
x_l(k+1) &= \sum_{j \in P} a_{lj} x_j(k) + h_l \\
&\leq \sum_{j \in P} \left(a_{lj}^+ \eta_j + a_{lj}^- \xi_j\right) + h_l \\
&< 0
\end{aligned}$$

for all $l \in N$ and $k \geq 0$. This clearly implies that for all $k \geq 0$, $\xi_i < x_i(k+1) < \eta_i$ if $i \in P$, and $\sigma(x_l(k+1)) = 0$ if $l \in N$, i.e., $x(k+1) \in D$. By the induction principle, D must be a local invariant set. Moreover, D is an activity invariant set. So the set of neurons with index P is an active invariant set of the network while the set of neurons with index N is an inactive invariant set.

Next, in the second part, we will prove that there is one stable equilibrium point located in D which exponentially attracts all the trajectories in D. Let $x(k, \bar{x}(0))$ and $x(k, \hat{x}(0))$ be two trajectories of the network (1) with initial conditions $\bar{x}(0)$ and $\hat{x}(0)$, respectively. Clearly, it holds that $x(k,\bar{x}(0)) \in D$ and $x(k,\hat{x}(0)) \in D$ for all $k \geq 0$ since D is a local invariant set. Then from (1), we have

$$x_i\left(k+1,\bar{x}(0)\right) = \sum_{j\in P} a_{ij} x_j\left(k,\bar{x}(0)\right) + h_i \tag{7}$$

and

$$x_i\left(k+1,\hat{x}(0)\right) = \sum_{j\in P} a_{ij} x_j\left(k,\hat{x}(0)\right) + h_i \tag{8}$$

for all $i = 1,2,\cdots,n$ and $k \geq 0$. Denote

$$z_i(k) = x_i\left(k,\bar{x}(0)\right) - x_i\left(k,\hat{x}(0)\right), \quad (i = 1,2,\cdots,n)$$

for $k \geq 0$. It follows from (7) and (8) that

$$z_i(k+1) = \sum_{j\in P} a_{ij} z_j(k), \quad (i = 1,2,\cdots,n) \tag{9}$$

for $k \geq 0$.

Consider a subsystem of (9)

$$z_i(k+1) = \sum_{j\in P} a_{ij} z_j(k), \quad (i \in P) \tag{10}$$

for $k \geq 0$.

By condition (5), we have

$$\sum_{j\in P}\left(a_{ij}^{+} - a_{ij}^{-}\right)(\eta_j - \xi_j) < (\eta_i - \xi_i), \quad (i \in P).$$

Thus, it immediately holds that

$$\frac{1}{\eta_i - \xi_i}\sum_{j\in P} |a_{ij}|(\eta_j - \xi_j) < 1, \quad (i \in P).$$

Denote by $\alpha_i = \eta_i - \xi_i > 0 (i \in P)$, then

$$\beta_i \triangleq \frac{1}{\alpha_i}\sum_{j\in P}\alpha_j |a_{ij}| < 1, \quad (i \in P).$$

Let $\beta = \max_{i\in P}\{\beta_i\}$. Now define functions

$$v_i(k) = \frac{|z_i(k)|}{\alpha_i}, \quad (i \in P)$$

for all $k \geq 0$. Then it follows from (10) that

$$\begin{aligned} v_i(k+1) &\leq \frac{1}{\alpha_i}\sum_{j\in P}\alpha_j |a_{ij}| \cdot v_j(k) \\ &\leq \frac{1}{\alpha_i}\sum_{j\in P}\alpha_j |a_{ij}| \cdot \|v(k)\| \\ &\leq \beta \cdot \|v(k)\| \\ &\leq \beta^2 \cdot \|v(k-1)\| \\ &\quad \vdots \\ &\leq \beta^{k+1} \cdot \|v(0)\| \\ &= e^{-(k+1)\ln\frac{1}{\beta}} \cdot \|v(0)\| \end{aligned}$$

for all $k \geq 0$ and $i \in P$. Since $\beta < 1$, it must holds that $\ln(1/\beta) > 0$. Let constant $\epsilon = \ln(1/\beta)$, then

$$v_i(k+1) \leq e^{-(k+1)\epsilon} \cdot \|v(0)\|.$$

Moreover, since

$$\|v(0)\| \leq \max_{i\in P}\left\{\frac{1}{\alpha_i}\right\} \cdot \|z(0)\| = \min_{i\in P}\{\alpha_i\} \cdot \|z(0)\|$$

it follows that

$$\begin{aligned} |z_i(k+1)| &\leq e^{-(k+1)\epsilon} \cdot \max_{i\in P}\{\alpha_i\} \cdot \min_{i\in P}\{\alpha_i\} \cdot \|z(0)\| \\ &= e^{-(k+1)\epsilon} \cdot \delta \cdot \|z(0)\| \end{aligned} \tag{11}$$

for all $k \geq 0$ and $i \in P$, where

$$\delta = \max_{i\in P}\{\alpha_i\} \cdot \min_{i\in P}\{\alpha_i\} > 0.$$

Next, we consider another subsystem of (9)

$$z_l(k+1) = \sum_{j\in P} a_{lj} z_j(k), \quad (l \in N)$$

for $k \geq 0$. It is clearly that

$$\begin{aligned} |z_l(k+1)| &\leq \max_{j\in P}\{z_j(k)\} \cdot \sum_{j\in P}|a_{lj}| \\ &\leq e^{-k\epsilon} \cdot \delta \cdot \|z(0)\| \cdot \sum_{j\in P}|a_{lj}| \end{aligned} \tag{12}$$

for $l \in N$ and $k \geq 0$. From (11) and (12), there must exist a constant $\bar{\delta} > 0$ such that

$$|z_i(k+1)| \leq \bar{\delta} \cdot \|z(0)\| \cdot e^{-(k+1)\epsilon}, \quad (i = 1,2,\cdots,n)$$

for $k \geq 0$. Then

$$\|x\left(k+1,\bar{x}(0)\right) - x\left(k+1,\hat{x}(0)\right)\| \leq \bar{\delta} \cdot \|\bar{x}(0) - \hat{x}(0)\| \cdot e^{-(k+1)\epsilon}$$

for all $k \geq 0$. By Lemma 1, this implies that there exists an equilibrium point in D which exponentially attracts all trajectories in D, i.e., D has an exponentially stable attractor. The proof is complete. ■

Given some division of neurons of the network (1), i.e., $P \cup N = \{1,2,\cdots,n\}$, and $P \cap N = \emptyset$, the theorem above shows that if there exists a pair of constant vector (ξ,η) such that (5) and (6), then the activity of each neuron in D keeps invariant. The location of D is indicated by (ξ,η). That is, the set of neurons with index P will keep active while the set of neurons with index N will keep inactive all the time, as long as the initial conditions belong to D.

Moreover, we have also shown that under the conditions of Theorem 1, the activity invariant set D has one exponentially stable attractor which is regarded as memory stored in the synaptic connections of the networks. Since the activity invariant set D is composed of two parts, active and inactive invariant set, each attractor has binary pattern. Furthermore, in the active invariant part, the neurons carry analogy information. Thus, the networks implement a form of hybrid analog-digital computation. In other words, the attractors of the network (1) could be used to store memories with both binary and analog information. Thus, it can provide new perspective to apply attractor networks for applications such as group winner-take-all, associative memory, etc.

Theorem 2: If there exist constants $0 < \xi_i < \eta_i$ $(i = 1,2,\cdots,n)$ such that

$$\begin{cases} \sum_{j\in P}\left(a_{ij}^{+}\xi_j + a_{ij}^{-}\eta_j\right) + h_i > \xi_i \\ \sum_{j\in P}\left(a_{ij}^{+}\eta_j + a_{ij}^{-}\xi_j\right) + h_i < \eta_i \end{cases}$$

for $i = 1, 2, \cdots, n$, then the set

$$D = \{x | x_i \in (\xi_i, \eta_i), (i = 1, 2, \cdots, n)\}$$

is an activity invariant set of the network (1). Moreover, D has an exponentially stable attractor.

Proof: Let $P = \{1, 2, \cdots, n\}$ and N be empty, the result follows from the proof of Theorem 1. ■

Theorem 3: If $h_i < 0$ $(i = 1, 2, \cdots, n)$, then the set

$$D = \{x | \sigma(x_i) = 0, (i = 1, 2, \cdots, n)\}$$

is an activity invariant set of the network (1). Moreover, D has an exponentially stable attractor.

Proof: Let $N = \{1, 2, \cdots, n\}$ and P be empty, the result follows from the proof of Theorem 1. ■

IV. Simulation Results

In this section, simulations will be carried out to show how to locate the activity invariant sets. A three-dimensional network will be employed for illustrations.

Let us consider the following three-dimensional network

$$\begin{cases} x_1(k+1) = 0.2\sigma(x_1(k)) - 3\sigma(x_2(k)) - 2\sigma(x_3(k)) + 1 \\ x_2(k+1) = -2\sigma(x_1(k)) + 0.2\sigma(x_2(k)) - 3\sigma(x_3(k)) + 1 \\ x_3(k+1) = -3\sigma(x_1(k)) - 4\sigma(x_2(k)) + 0.2\sigma(x_3(k)) + 1 \end{cases} \quad (13)$$

Clearly, $a_{ii} = 0.2$ ($i = 1, 2, 3$), $a_{12} = a_{31} = a_{23} = -3$, $a_{13} = a_{21} = -2$, $a_{32} = -4$ and $h_i = 1$ ($i = 1, 2, 3$).

Taking $P = \{1\}, N = \{2, 3\}$, by conditions (5) and (6) of Theorem 1, we have inequalities for possible invariant sets as

$$\begin{cases} 0.2\xi_1 + 1 > \xi_1 \\ 0.2\eta_1 + 1 < \eta_1 \\ -2\xi_1 + 1 < 0 \\ -3\xi_1 + 1 < 0 \\ 0 < \xi_1 < \eta_1. \end{cases}$$

Solving the inequalities, one can get a solution that $\xi_1 = 1$, $\eta_1 = 2$. Thus

$$D_1 = \{x | 1 < x_1 < 2; \quad x_2 < 0; \quad x_3 < 0\}$$

is an activity invariant set, and the neuron with index $i = 1$ is active invariant in D_1 while the neurons with index $i = 2, 3$ are both inactive invariant in D_1. Moreover, by Theorem 1, D_1 has an exponentially stable attractor.

Taking $P = \{2\}, N = \{1, 3\}$, solve the inequalities

$$\begin{cases} 0.2\xi_2 + 1 > \xi_2 \\ 0.2\eta_2 + 1 < \eta_2 \\ -3\xi_2 + 1 < 0 \\ -4\xi_2 + 1 < 0 \\ 0 < \xi_2 < \eta_2. \end{cases}$$

It can be solved that $\xi_2 = 1$, $\eta_2 = 2$. Thus

$$D_2 = \{x | x_1 < 0; \quad 1 < x_2 < 2; \quad x_3 < 0\}$$

is an activity invariant set and the neuron with index $i = 2$ is active invariant in D_2 while the neurons with index $i = 1,3$ are both inactive invariant in D_2. Moreover, by Theorem 1, D_2 also has an exponentially stable attractor.

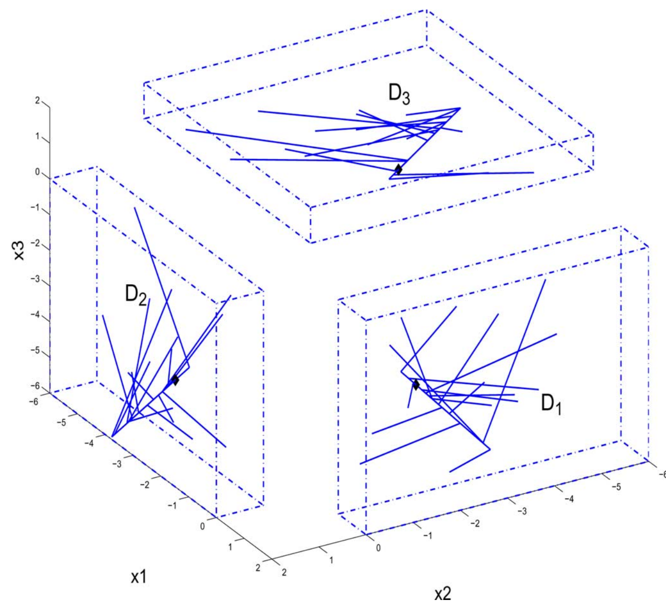

Fig. 2. Activity invariant sets and exponentially stable attractors of the network (13). There are three local stable equilibrium points $(1.25, -1.5, -2.75)^T$, $(-2.75, 1.25, -4)^T$ and $(-1.5, -2.75, 1.25)^T$ located in the activity invariant sets D_1, D_2 and D_3, respectively.

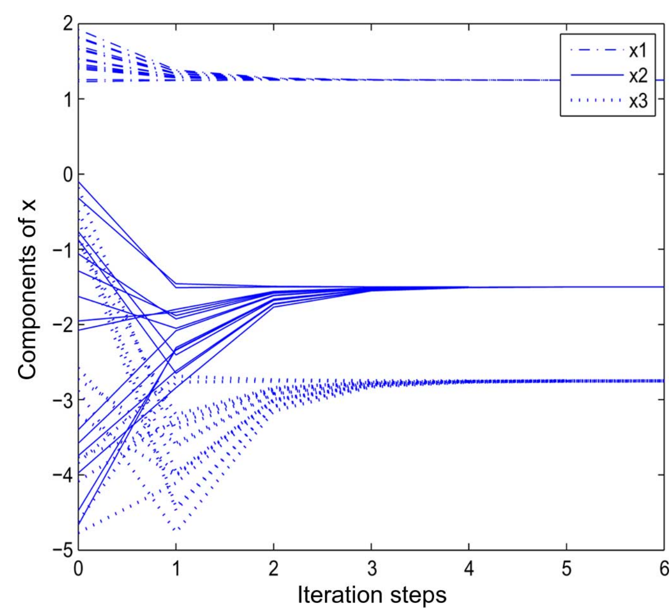

Fig. 3. Convergence of the network (13). The trajectories start from 15 randomly initial points located in D_1. They all exponentially converge to the stable equilibrium point $x^* = (1.25, -1.5, -2.75)^T \in D_1$.

Taking $P = \{3\}, N = \{1, 2\}$ and solving the inequalities

$$\begin{cases} 0.2\xi_3 + 1 > \xi_3 \\ 0.2\eta_3 + 1 < \eta_3 \\ -2\xi_3 + 1 < 0 \\ -3\xi_3 + 1 < 0 \\ 0 < \xi_3 < \eta_3, \end{cases}$$

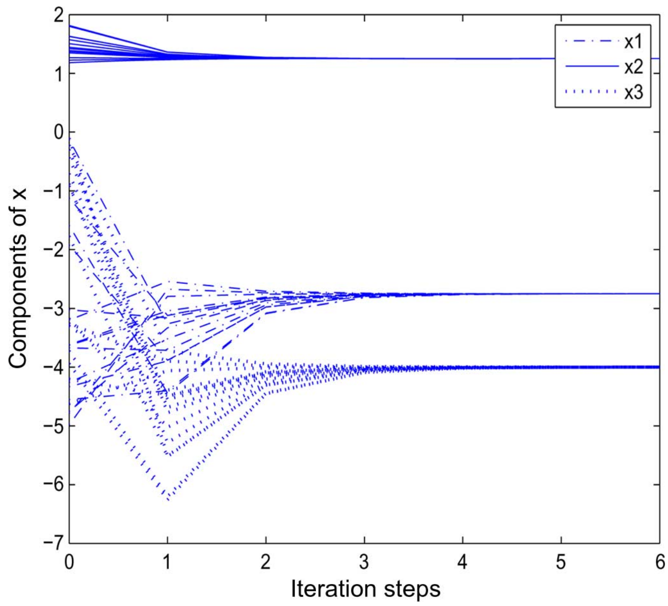

Fig. 4. Convergence of the network (13). The trajectories start from 15 randomly initial points located in D_2. They all exponentially converge to the stable equilibrium point $x^* = (-2.75, 1.25, -4)^T \in D_2$.

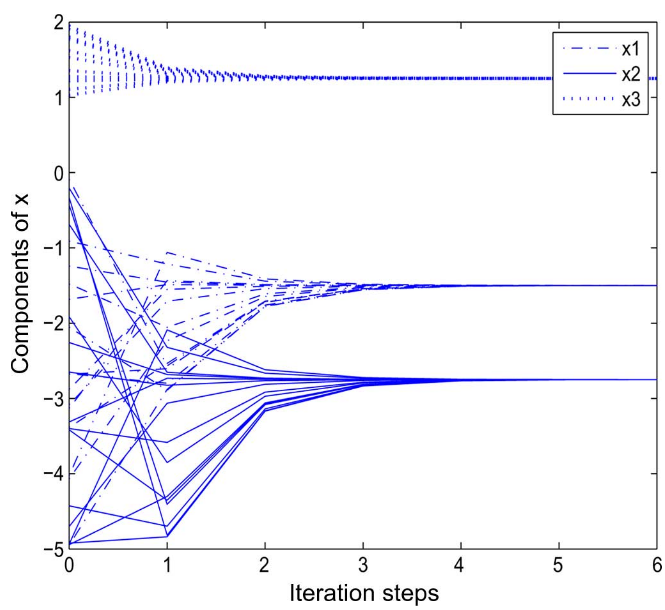

Fig. 5. Convergence of the network (13). The trajectories start from 15 randomly initial points located in D_3. They all exponentially converge to the stable equilibrium point $x^* = (-1.5, -2.75, 1.25)^T \in D_3$.

one can get a solution that $\xi_3 = 1, \eta_3 = 2$. Thus

$$D_3 = \{x | x_1 < 0; \quad x_2 < 0; \quad 1 < x_3 < 2\}$$

is an activity invariant set and the neuron with index $i = 3$ is active invariant in D_3 while the neurons with index $i = 1,2$ are both inactive invariant in D_3. In the same way, by Theorem 1, D_3 has an exponentially stable attractor.

We also considered the cases: $P = \{1, 2\}$, $N = \{3\}$; $P = \{1, 3\}$, $N = \{2\}$; $P = \{2, 3\}$, $N = \{1\}$; $P = \{1, 2, 3\}$ with N to be empty; $N = \{1, 2, 3\}$ with P to be empty. However, we have not found solutions for inequalities (5) and (6). Computer simulations also have not observed other exponentially stable attractors different from the above three ones.

Fig. 2 shows the activity invariant sets and the exponentially stable attractors of the network (13). There are total three local stable equilibrium points (drew by diamond symbol) $(1.25, -1.5, -2.75)^T$, $(-2.75, 1.25, -4)^T$ and $(-1.5, -2.75, 1.25)^T$ located in the activity invariant sets D_1, D_2 and D_3, respectively, and attract all trajectories in the corresponding regions. Figs. 3–5 show the convergence of the trajectories starting from the 45 randomly initial points located in D_1, D_2 and D_3, respectively.

V. Conclusion

In this technical note, activity invariant sets and exponentially stable attractors have been studied. Conditions have been derived to locate the activity invariant sets. Furthermore, it shows that under some conditions, an activity invariant set can possess an exponentially stable attractor. Such an attractor carries both binary and analog information. We believe these interesting properties can give new perspective for application of attractor networks to group winner-take-all, associative memory, etc. More research in this direction will be carried out in the future.

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$\mathcal{H}_\infty$ Filtering of Discrete-Time Markov Jump Linear Systems Through Linear Matrix Inequalities

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***Abstract*—This technical note addresses the discrete-time Markov jump linear systems $\mathcal{H}_\infty$ filtering design problem. First, under the assumption that the Markov parameter is measurable, the main contribution is the linear matrix inequality (LMI) characterization of all linear filters such that the estimation error remains bounded by a given $\mathcal{H}_\infty$ norm level, yielding the complete solution of the mode-dependent filtering design problem. Based on this result, a robust filter design able to deal with polytopic uncertainty is considered. Second, from the same LMI characterization, a design procedure for mode-independent filtering is proposed. Some examples are solved for illustration and comparisons.**

***Index Terms*—Discrete-time systems, linear matrix inequalities (LMIs), Markov jump linear systems, robust filtering.**

I. Introduction

Dynamic systems that present sudden changes on their structures or parameters have been the subject of several studies in the last decades. Among the different ways to model such a dynamic system, one of increasing interest is the Markov jump linear system (MJLS). One of the first works in the literature dealing with this class of models was presented in [1]. After that, a large amount of theory and design procedures have been developed in order to extend concepts of the deterministic discrete-time systems to this special class, namely stability and testable conditions [10], [11]; optimal state feedback control [9]; state feedback $\mathcal{H}_2$ optimization via convex programming [4]; state feedback $\mathcal{H}_2$ optimization via linear matrix inequalities (LMIs) [8], state feedback $\mathcal{H}_\infty$ optimization via LMIs [12] and $\mathcal{H}_2$ filtering [5], just to stay with a few of them.

An important assumption to consider for the MJLS filtering design problem is if the Markov chain state, also known as mode, is available to the filter at every instant of time $k \in \mathbb{N}$. Based on this characteristic, the design is said to be either mode-dependent or mode-independent.

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The problem of determining a strictly proper optimal $\mathcal{H}_\infty$ mode-dependent filter was solved in [7] with the use of LMIs. In that paper, only the observer-based filter gains are obtained and the particular filter structure based on the internal model of the plant makes the result useless for robust filter design. For two dimensional discrete-time MJLS, [15] gives sufficient conditions for $\mathcal{H}_\infty$ filtering design. Considering the mode-independent assumption, [6] proposes sufficient conditions that impose the estimation error $\mathcal{H}_\infty$ norm a guaranteed level, once again only for strictly proper filters. In this technical note, the set of all full order linear mode-dependent proper filters with bounded estimation error $\mathcal{H}_\infty$ norm is obtained. This set is expressed in terms of LMIs, allowing the optimal $\mathcal{H}_\infty$ filtering design problem to be solved in one shot rather than iteratively. When the filter is constrained to be strictly proper, the one proposed in [7] is obtained as a particular case. The use of LMIs also allows us to include additional constraints to the basic problem, as for instance to design robust filters able to face parameter uncertainty. Another additional constraint to the main problem allows us to design mode-independent filters and to compare our results with those provided in [6]. Moreover, we believe that the necessary and sufficient conditions reported here can be useful to improve the results of [15].

The notation is standard. For real matrices or vectors $(')$ indicates transpose. For the sake of easing the notation of partitioned symmetric matrices, the symbol $(\bullet)$ denotes generically each of its symmetric blocks. The set of nonnegative integers is denoted by $\mathbb{N}$. The set of Markov chain states is $\mathbb{K} = \{1, 2, \cdots, N\}$. Given N^2 nonnegative real numbers p_{ij} satisfying $\sum_{j=1}^{N} p_{ij} = 1, i \in \mathbb{K}$ and N real matrices $X_j, j \in \mathbb{K}$, the convex combination of these matrices with weights p_{ij} is denoted by $X_{pi} = \sum_{j=1}^{N} p_{ij} X_j$ for $i \in \mathbb{K}$. The symbol $\mathcal{E}\{\cdot\}$ denotes mathematical expectation of $\{\cdot\}$. For any stochastic signal $z(k)$, defined in the discrete-time domain $k \in \mathbb{N}$, the quantity $\|z\|_2^2 = \sum_{k=0}^{\infty} \mathcal{E}\{z(k)' z(k)\}$ is its squared norm. The set of signals $z(k) \in \mathbb{R}^r$, defined for all $k \in \mathbb{N}$, such that $\|z\|_2^2 < \infty$ is denoted $\mathcal{L}_2^r$.

II. Problem Formulation

A discrete-time Markov jump linear system (MJLS) is described by the following stochastic equations:

$$\mathbb{G} : \begin{cases} x(k+1) = A(\theta_k)x(k) + J(\theta_k)w(k) \\ z(k) = C_z(\theta_k)x(k) + E_z(\theta_k)w(k) \\ y(k) = C_y(\theta_k)x(k) + E_y(\theta_k)w(k) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^m$ is the external perturbation, $z(k) \in \mathbb{R}^r$ is the output to be estimated and $y(k) \in \mathbb{R}^q$ is the measured output. It is assumed that the system evolves from $x(0) = 0$. The state space matrices (1) depend on a Markov chain taking values in the finite set $\mathbb{K}$ with the associated transition probability matrix $\mathbb{P} \in \mathbb{R}^{N \times N}$ whose elements are given by $p_{ij} = \text{Prob}(\theta_{k+1} = j | \theta_k = i)$ which clearly satisfies the normalized constraints $p_{ij} \geq 0$ and $\sum_{j=1}^{N} p_{ij} = 1$ for each $i \in \mathbb{K}$. To ease the presentation, the following notations $A(\theta_k) := A_i$, $J(\theta_k) := J_i$, $C_z(\theta_k) := C_{zi}$, $E_z(\theta_k) := E_{zi}$, $C_y(\theta_k) := C_{yi}$ and $E_y(\theta_k) := E_{yi}$ are used whenever $\theta_k = i \in \mathbb{K}$. An important concept related to the model (1) is its $\mathcal{H}_\infty$ norm denoted as $\|\mathbb{G}\|_\infty$ and formally characterized by means of the next definition, based on a well known property of system (1) which states that if it is stable (SMS—Second Moment Stable) and $w \in \mathcal{L}_2^m$ then $z \in \mathcal{L}_2^r$ (cf. Proposition 2 from [3]).

Definition 1: Assume that $\mathbb{G}$ is stable. The $\mathcal{H}_\infty$ norm of the system $\mathbb{G}$ from the input w to the output z is given by

$$\|\mathbb{G}\|_\infty^2 = \sup_{0 \neq w \in \mathcal{L}_2^m, \theta_0 \in \mathbb{K}} \frac{\|z\|_2^2}{\|w\|_2^2}. \tag{2}$$