

Stability of Neutral Lotka–Volterra Systems

Zhang Yi*

Institute of Mathematics, Academia Sinica, Beijing 100080, People's Republic of China

Submitted by J. Eisenfeld

Received April 12, 1994

This paper studies the stability of neutral Lotka–Volterra systems with bounded delay and unbounded delay, respectively. Sufficient conditions for stability are given in terms of systems parameters. © 1996 Academic Press, Inc.

1. INTRODUCTION

The stability theory of Lotka–Volterra systems with delay has been considerably developed during the past years and various interesting results have been reported. However, not much has been developed in the direction of neutral Lotka–Volterra systems except a few reports [1–4] in which simple systems are considered. The purpose of this paper is to investigate sufficient conditions for stability of more general neutral Lotka–Volterra systems.

We consider neutral Lotka–Volterra systems with bounded delay and unbounded delay, respectively. In the case of bounded delay, a uniform asymptotic stability result will be derived. When dropping the assumption of boundedness of the delay, we discuss asymptotic stability which is relatively weaker than uniform asymptotic stability.

2. PRELIMINARIES

Consider the neutral Lotka–Volterra system

$$\begin{aligned} \dot{x}_i(t) &= [x_i(t) + x_i^*] \\ &\cdot \sum_{j=1}^n [a_{ij}x_j(t) + b_{ij}x_j(t - \tau_1(t)) + c_{ij}\dot{x}_j(t - \tau_2(t))], \\ &t \geq t_* \quad (i = 1, 2, \dots, n), \end{aligned} \quad (2.1)$$

* Present address: Department of Mathematics, University of Electronic Science and Technology of China, Chengdu 610054, People's Republic of China.

where $x_i^* > 0$, a_{ij} , b_{ij} , c_{ij} are constants, and $\tau_l(t)$ ($l = 1, 2$) are nonnegative continuous functions.

For any $t_0 \geq t_*$, let

$$Et_0 = \{t - \tau_l(t) | t - \tau_l(t) \leq t_0, t \geq t_0 \ (l = 1, 2)\} \cup \{t_0\}.$$

We assume the initial condition is

$$\begin{cases} x_i(t) = \varphi_i(t), & t \in E_{t_0}, \\ \dot{x}_i(t) = \dot{\varphi}_i(t), & t \in E_{t_0}, \\ \varphi_i(t_0) + x_i^* > 0 & (i = 1, \dots, n), \end{cases} \quad (2.2)$$

where $\varphi_i(t)$, $\dot{\varphi}_i(t)$ ($i = 1, \dots, n$) are continuous and bounded functions defined on E_{t_0} . Obviously, with the initial condition (2.2), every solution of (2.1) satisfies

$$x_i(t) + x_i^* > 0 \quad (i = 1, \dots, n),$$

for $t \geq t_0$. In fact, we have from (2.1) that

$$x_i(t) + x_i^* = [\varphi_i(t_0) + x_i^*] e^{\sum_{j=1}^n \int_{t_0}^t [a_{ij}x_j(s) + b_{ij}x_j(s - \tau_1(s)) + c_{ij}\dot{x}_j(s - \tau_2(s))] ds}.$$

Then, the result follows from (2.2).

Let

$$\|\varphi\| = \max_{1 \leq i \leq n} \left[\sup_{t \in E_{t_0}} |\varphi_i(t)| \right],$$

and

$$\|\dot{\varphi}\| = \max_{1 \leq i \leq n} \left[\sup_{t \in E_{t_0}} |\dot{\varphi}_i(t)| \right].$$

DEFINITION 2.1. The zero solution of (2.1) is said to be stable, if for any $t_0 \geq t_*$ and any $\varepsilon > 0$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$\|\varphi\| < \delta(t_0, \varepsilon), \quad \|\dot{\varphi}\| < \delta(t_0, \varepsilon)$$

imply

$$|x_i(t)| < \varepsilon, \quad |\dot{x}_i(t)| < \varepsilon \quad (i = 1, 2, \dots, n)$$

for $t \geq t_0$. If $\delta(t_0, \varepsilon)$ is independent of t_0 , i.e., $\delta(t_0, \varepsilon) = \delta(\varepsilon)$, then the zero solution of (2.1) is said to be uniformly stable.

DEFINITION 2.2. If the zero solution of (2.1) is stable, and for each $t_0 \geq t_*$, there is a $\delta_0 = \delta(t_0) > 0$ such that

$$\|\varphi\| < \delta_0, \quad \|\dot{\varphi}\| < \delta_0$$

imply

$$|x_i(t)| \rightarrow 0, \quad |\dot{x}_i(t)| \rightarrow 0 \quad (i = 1, \dots, n)$$

as $t \rightarrow +\infty$, then the zero solution of (2.1) is said to be asymptotically stable. If the zero solution of (2.1) is uniformly stable, and there exists a $\eta > 0$, for any $\gamma > 0$ there is a $T = T(\gamma) > 0$ such that $\|\varphi\| < \eta$, $\|\dot{\varphi}\| < \eta$ imply

$$|x_i(t)| < \gamma, \quad |\dot{x}_i(t)| < \gamma \quad (i = 1, \dots, n)$$

for $t \geq t_0 + T$. Then, the zero solution of (2.1) is said to be uniformly asymptotically stable.

Next, we make the following assumptions.

$$(H_1) \quad a_{ii} < 0 \quad (i = 1, \dots, n).$$

$$(H_2) \quad \rho(\Omega) < 1, \text{ where } \rho(\Omega) \text{ denotes the spectral radius of } \Omega, \text{ and}$$

$$\Omega \triangleq (\omega_{ij})_{2n \times 2n} \triangleq \begin{pmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{pmatrix}$$

with

$$\Omega_1 = \left(\frac{|a_{ij}|(1 - \delta_{ij}) + |b_{ij}|}{|a_{ii}|} \right)_{n \times n}, \quad \Omega_2 = \left(\frac{|c_{ij}|}{|a_{ii}|} \right)_{n \times n},$$

$$\Omega_3 = [x_i^* (|a_{ij}| + |b_{ij}|)]_{n \times n}, \quad \Omega_4 = (x_i^* |c_{ij}|)_{n \times n},$$

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \quad (i, j = 1, \dots, n). \end{cases}$$

$$(H_3) \quad \text{There is a constant } \tau > 0 \text{ such that}$$

$$0 \leq \tau_l(t) \leq \tau \quad (l = 1, 2)$$

for $t \geq t_*$.

$$(H_4) \quad t - \tau_l(t) \rightarrow +\infty \text{ as } t \rightarrow +\infty \quad (l = 1, 2).$$

3. MAIN RESULTS

THEOREM 3.1. If (H_1) , (H_2) , and (H_3) hold, then the zero solution of (2.1) is uniformly asymptotically stable.

Proof. Since $\rho(\Omega) < 1$, there exist constants $\alpha_1 > 0, \dots, \alpha_{2n} > 0$ such that

$$\frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \omega_{lh} < 1 \quad (l = 1, \dots, 2n). \quad (3.1)$$

Let

$$S_l(t) = \begin{cases} \frac{|x_l(t)|}{\alpha_l}, & 1 \leq l \leq n \\ \frac{|\dot{x}_{l-h}(t)|}{\alpha_l}, & n+1 \leq l \leq 2n, t \in E_{t_0} \cup [t_0, +\infty). \end{cases}$$

Then, for any $t_0 \geq t_*$, we have from (2.1) that

$$\begin{aligned} D^+ S_l(t) \leq [x_l(t) + x_l^*] & \left[a_{ll} S_l(t) + \frac{1}{\alpha_l} \sum_{j=1}^n \alpha_j (|a_{lj}| (1 - \delta_{lj}) S_j(t) \right. \\ & \left. + |b_{lj}| \cdot S_j(t - \tau_1(t))) \right. \\ & \left. + \frac{1}{\alpha_l} \sum_{j=1}^n \alpha_j |c_{lj}| \cdot S_{j+n}(t - \tau_2(t)) \right], \end{aligned} \quad 1 \leq l \leq n \quad (3.2)$$

and

$$\begin{aligned} S_l(t) \leq & \left[\frac{S_{l-n}(t) \alpha_{l-n}}{x_{l-n}^*} + 1 \right] \\ & \cdot \frac{1}{\alpha_l} \sum_{j=1}^n \left[\alpha_j x_{l-n}^* (|a_{l-n,j}| S_j(t) + |b_{l-n,j}| \cdot S_j(t - \tau_1(t))) \right. \\ & \left. + \alpha_{j+n} x_{l-n}^* |c_{l-n,j}| \cdot S_{j+n}(t - \tau_2(t)) \right], \quad n+1 \leq l \leq 2n \quad (3.3) \end{aligned}$$

for $t \geq t_0$.

By (3.1), we can choose a constant $m > 0$ such that

$$\begin{cases} m < \frac{x_l^*}{\alpha_l}, & 1 \leq l \leq n \\ \left(\frac{\alpha_{l-n} \cdot m}{x_{l-n}^*} + 1 \right) \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \omega_{lh} < 1, & n+1 \leq l \leq 2n. \end{cases} \quad (3.4)$$

To any $\varepsilon > 0$ ($\varepsilon \leq \max_{1 \leq h \leq 2n} (\alpha_h) \cdot m$), choose

$$\delta = \min_{1 \leq h \leq 2n} (\alpha_h) \frac{1}{\max_{1 \leq h \leq 2n} (\alpha_h)} \cdot \varepsilon.$$

Now, we will prove that for any $t_0 \geq t_*$, $\|\varphi\| < \delta$, $\|\dot{\varphi}\| < \delta$ imply that

$$S_l(t) < \frac{1}{\max_{1 \leq h \leq 2n} (\alpha_h)} \cdot \varepsilon \triangleq \varepsilon_0, \quad 1 \leq l \leq 2n \tag{3.5}$$

for $t \geq t_0$. In fact, if (3.5) does not hold, there must exist $t_1 > t_0$ and some l such that

$$S_l(t_1) = \varepsilon_0, \quad S_h(t) \begin{cases} < \varepsilon_0, t \in E_{t_0} \cup [t_0, t_1), h = l \\ \leq \varepsilon_0, t \in E_{t_0} \cup [t_0, t_1], h \neq l. \end{cases} \tag{3.6}$$

This follows from the fact that $S_h(t) < \varepsilon_0$ ($h = 1, \dots, 2n$) for $t \in E_{t_0}$.

Case 1. $1 \leq l \leq n$. Then $D^+ S_l(t_1) \geq 0$. However, from (3.2) and (3.1) we have

$$D^+ S_l(t_1) \leq -[x_l(t_1) + x_l^*] \cdot |a_{ll}| \cdot \left(1 - \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \cdot \omega_{lh} \right) \cdot \varepsilon_0 < 0.$$

This yields a contradiction.

Case 2. $n + 1 \leq l \leq 2n$. It follows from (3.3) and (3.4) that

$$\begin{aligned} S_l(t_1) &\leq \left(\frac{\alpha_{l-n} \cdot \varepsilon_0}{x_{l-n}^*} + 1 \right) \cdot \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \omega_{lh} \cdot \varepsilon_0 \\ &\leq \left(\frac{\alpha_{l-n} \cdot m}{x_{l-n}^*} + 1 \right) \cdot \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \omega_h \cdot \varepsilon_0 \\ &< \varepsilon_0, \end{aligned}$$

which is a contradiction to (3.6). Thus, (3.5) holds.

Hence, from (3.5), $\|\varphi\| < \delta$, $\|\dot{\varphi}\| < \delta$ imply that

$$|x_i(t)| < \varepsilon, \quad |\dot{x}_i(t)| < \varepsilon \quad (i = 1, \dots, n)$$

for $t \geq t_0$. This shows that the zero solution of (2.1) is uniformly stable.

Now, if in the above, we take $\varepsilon = \max_{1 \leq h \leq 2n} (\alpha_h) \cdot m$ and let

$$\delta_0 = \min_{1 \leq h \leq 2n} (\alpha_h) \cdot m,$$

then we have that $\|\varphi\| < \delta_0$, $\|\dot{\varphi}\| < \delta_0$ imply

$$S_l(t) < m \quad (l = 1, \dots, 2n)$$

for $t \geq t_0$.

We shall prove for any $\gamma > 0$, there exists a $T(\gamma) > 0$ such that $\|\varphi\| < \delta_0$, $\|\dot{\varphi}\| < \delta_0$ imply

$$|x_i(t)| < \gamma, \quad |\dot{x}_i(t)| < \gamma \quad (i = 1, \dots, n)$$

for $t \geq t_0 + T$. Define $\gamma_0 \triangleq (1/\max_{1 \leq h \leq 2n}(\alpha_h)) \cdot \gamma$, and choose η such that

$$\left\{ \begin{array}{l} 0 < \eta < \frac{1 - (1/\alpha_l) \sum_{h=1}^{2n} \alpha_h \cdot \omega_{lh}}{(1/\alpha_l) \sum_{h=1}^{2n} \alpha_h \cdot \omega_{lh}} \cdot \gamma_0, \\ 1 \leq l \leq n \\ 0 < \eta < \frac{\gamma_0}{((\alpha_{l-n} \cdot m)/x_{l-n}^* + 1)(1/\alpha_l) \sum_{h=1}^{2n} \alpha_h \cdot \omega_{lh}} - \gamma_0, \\ n + 1 \leq l \leq 2n. \end{array} \right. \quad (3.7)$$

Let N be the first nonnegative integer such that

$$\gamma_0 + N \cdot \eta \geq m$$

and take

$$t_k = t_0 + kT^*, \quad k = 0, 1, 2, \dots, N,$$

where

$$T^* = \tau + \frac{2 \cdot \max_{1 \leq i \leq n} [\ln((\alpha_i m + x_i^*)/x_i^*), |\ln((x_i^* - \alpha_i m)/x_i^*)|]}{\min_{1 \leq i \leq n} \left[\alpha_i |a_{ii}| \left(\gamma_0 \left(1 - (1/\alpha_i) \sum_{h=1}^{2n} \alpha_h \omega_{ih} \right) - (1/\alpha_i) \sum_{h=1}^{2n} \alpha_h \cdot \omega_{ih} \cdot \eta \right) \right]}$$

Next, we prove $\|\varphi\| < \delta_0$, $\|\dot{\varphi}\| < \delta_0$ imply that

$$S_l(t) \leq \gamma_0 + (N - k)\eta \quad (l = 1, \dots, 2n) \quad (3.8)$$

for all $t \geq t_k$, $k = 0, 1, 2, \dots, N$.

If $k = 0$, obviously, (3.8) holds. Suppose that (3.8) holds for some k , $0 \leq k < N$. We will show that (3.8) hold for $k + 1$, i.e.,

$$S_l(t) \leq \gamma_0 + (N - k - 1)\eta \quad (l = 1, \dots, 2n) \quad (3.9)$$

for $t \geq t_{k+1}$. To prove (3.9), we first prove that there exist $\bar{t} \in [t_k + \tau, t_{k+1}]$

such that

$$S_l(\bar{t}) \leq \gamma_0 + (N - k - 1)\eta \quad (l = 1, \dots, 2n). \quad (3.10)$$

In fact, if (3.10) does not hold, there must exist some l such that

$$S_l(t) > \gamma_0 + (N - k - 1)\eta$$

for all $t \in [t_k + \tau, t_{k+1}]$. Hence,

$$\begin{aligned} \sup_{t-\tau \leq \theta \leq t} S_h(\theta) &\leq \gamma_0 + (N - k)\eta = \gamma_0 + (N - k - 1)\eta + \eta \\ &< S_l(t) + \eta \quad (h = 1, \dots, 2n) \end{aligned}$$

for all $t \in [t_k + \tau, t_{k+1}]$.

Case 1. $1 \leq l \leq n$. We have from (2.1) that

$$\begin{aligned} D^+ \left| \ln \frac{x_l(t) + x_l^*}{x_l^*} \right| &\leq a_{ll} |x_l(t)| + \sum_{j=1}^n \left[|a_{lj}|(1 - \delta_{lj}) |x_j(t)| \right. \\ &\quad \left. + |b_{lj}| \cdot |x_j(t - \tau_1(t))| + |c_{lj}| \cdot |\dot{x}_j(t - \tau_2(t))| \right] \\ &\leq -\alpha_l |a_{ll}| \left[S_l(t) - \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \cdot \omega_{lh} (S_l(t) + \eta) \right] \\ &\leq -\alpha_l |a_{ll}| \left[S_l(t) \left(1 - \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \omega_{lh} \right) - \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \cdot \omega_{lh} \cdot \eta \right] \\ &\leq -\alpha_l |a_{ll}| \left[\gamma_0 \left(1 - \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \omega_{lh} \right) - \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \cdot \omega_{lh} \cdot \eta \right] \end{aligned}$$

for $t \in [t_k + \tau, t_{k+1}]$, and so

$$\begin{aligned} \left| \ln \frac{x_l(t_{k+1}) + x_l^*}{x_l^*} \right| &\leq \left| \ln \frac{x_l(t_k + \tau) + x_l^*}{x_l^*} \right| \\ &\quad - \alpha_l |a_{ll}| \left[\gamma_0 \left(1 - \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \omega_{lh} \right) \right. \\ &\quad \left. - \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \omega_{lh} \cdot \eta \right] (T^* - \tau) \\ &\leq -\max \left[\ln \frac{\alpha_l m + x_l^*}{x_l^*}, \left| \ln \frac{x_l^* - \alpha_l m}{x_l^*} \right| \right] < 0 \end{aligned}$$

which is a contradiction.

Case 2. $n + 1 \leq l \leq 2n$. From (3.3) and (3.7), we have for $t \in [t_k + \tau, t_{k+1}]$ that

$$\begin{aligned} S_l(t) &\leq \left[\frac{m\alpha_{l-n}}{x_{l-n}^*} + 1 \right] \cdot \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \cdot \omega_{lh} \cdot (S_l(t) + \eta) \\ &\leq \left[\frac{m\alpha_{l-n}}{x_{l-n}^*} + 1 \right] \cdot \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \omega_{lh} \left(1 + \frac{\eta}{\gamma_0} \right) \cdot S_l(t) \\ &< S_l(t). \end{aligned}$$

This is also a contradiction, and (3.10) follows. Next, we claim

$$S_l(t) \leq \gamma_0 + (N - k - 1)\eta \quad (l = 1, \dots, 2n) \quad (3.11)$$

for all $t \geq \bar{i}$. Otherwise, there must exist $\hat{t} > \bar{i}$ and some l such that

$$S_l(\hat{t}) > \gamma_0 + (N - k - 1)\eta$$

and $D^+ S_l(\hat{t}) > 0$ if $1 \leq l \leq n$. Thus,

$$\sup_{\hat{i} - \tau \leq t \leq \hat{i}} S_h(t) \leq \gamma_0 + (N - k)\eta < S_l(\hat{t}) + \eta \quad (h = 1, \dots, 2n).$$

If $1 \leq l \leq n$, we have from (3.2) that

$$\begin{aligned} D^+ S_l(\hat{t}) &\leq -[x_l(\hat{t}) + x_l^*] \cdot |a_{ll}| \\ &\quad \cdot \left[S_l(\hat{t}) \left(1 - \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \omega_{lh} \right) - \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \omega_{lh} \cdot \eta \right] \\ &\leq -[x_l(\hat{t}) + x_l^*] \cdot |a_{ll}| \\ &\quad \cdot \left[\gamma_0 \left(1 - \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \omega_{lh} \right) - \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \omega_{lh} \cdot \eta \right] \\ &< 0, \end{aligned}$$

which is a contradiction. If $n + 1 \leq l \leq 2n$, then by a proof similar to the one used in Case 2 to establish (3.10), we again reach a contradiction. Therefore, (3.11) holds.

Note that $t_{k+1} \geq \bar{i}$ and thus (3.9) holds. By induction, (3.8) follows.

Taking $k = N$ in (3.8), we obtain

$$S_l(t) \leq \gamma_0 \quad (l = 1, \dots, 2n)$$

for $t \geq t_0 + T$, where $T = NT^*$, and so

$$|x_i(t)| < \gamma, \quad |\dot{x}_i(t)| < \gamma \quad (i = 1, \dots, n)$$

for $t \geq t_0 + T$. Obviously, T is independent of t_0 . The proof is now complete.

THEOREM 3.2. *If (H_1) , (H_2) , and (H_4) hold, then the zero solution of (2.1) is asymptotically stable.*

Proof. By a slight modification of the proof of the first part of Theorem 3.1, it is easy to prove that the zero solution of (2.1) is stable. Moreover, we can choose

$$\delta_0 = \min_{1 \leq h \leq 2n} (\alpha_h) \cdot m$$

such that $\|\varphi\| < \delta_0, \|\dot{\varphi}\| < \delta_0$ imply

$$S_h(t) < m \quad (h = 1, \dots, 2n)$$

for $t \geq t_0$. Next, we will prove that $\|\varphi\| < \delta_0, \|\dot{\varphi}\| < \delta_0$ imply

$$|x_i(t)| \rightarrow 0, \quad |\dot{x}_i(t)| \rightarrow 0 \quad (i = 1, \dots, n)$$

as $t \rightarrow +\infty$. Let

$$\limsup_{t \rightarrow +\infty} S_h(t) = \sigma_h \quad (h = 1, \dots, 2n).$$

Obviously, $0 \leq \sigma_h \leq m$ ($h = 1, \dots, 2n$). Suppose $\sigma_l = \max_{1 \leq h \leq 2n} \{\sigma_h\}$, we first prove $\sigma_l = 0$. If $\sigma_l \neq 0$, i.e., $\sigma_l > 0$, we choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \frac{1 - (1/\alpha_l) \sum_{h=1}^{2n} \alpha_h \cdot \omega_{lh}}{1 + (1/\alpha_l) \sum_{h=1}^{2n} \alpha_h \cdot \omega_{lh}} \cdot \sigma_l. \tag{3.12}$$

By (H_4) , there is a $T_1 \geq t_0$ such that $t \geq T_1$ implies

$$S_h(t - \tau_\rho(t)) \leq \sigma_h + \varepsilon \leq \sigma_l + \varepsilon, \quad 1 \leq h \leq 2n \quad (\rho = 1, 2).$$

Case 1. $n + 1 \leq l \leq 2n$. From (3.3), we have

$$S_l(t) \leq \left[\frac{\alpha_{l-n} \cdot m}{x_{l-n}^*} + 1 \right] \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \omega_{lh} \cdot (\sigma_l + \varepsilon).$$

Letting $t \rightarrow +\infty$, it follows that

$$\sigma_l \leq \left[\frac{\alpha_{l-n} m}{x_{l-n}^*} + 1 \right] \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \cdot \omega_{lh} \cdot (\sigma_l + \varepsilon).$$

Letting $\varepsilon \rightarrow 0$, we obtain

$$\begin{aligned} \sigma_l &\leq \left[\frac{\alpha_{l-n} \cdot m}{x_{l-n}^*} + 1 \right] \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \cdot \omega_{lh} \cdot \sigma_l \\ &< \sigma_l \end{aligned}$$

which is a contradiction.

Case 2. $1 \leq l \leq n$. We first claim that there exists $T_2 \geq T_1$ such that

$$D^+S_l(t) < 0$$

for all $t \geq T_2$. Otherwise, there is a $T_3 > T_2$ such that

$$D^+S_l(T_3) \geq 0, \quad S_l(T_3) \geq \sigma_l - \varepsilon.$$

From (3.2) and (3.12), we have

$$\begin{aligned} D^+S_l(T_3) &\leq -[x_l(T_3) + x_l^*] \cdot |a_{ll}| \\ &\quad \cdot \left[(\sigma_k - \varepsilon) - \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \cdot \omega_{lh} \cdot (\sigma_l + \varepsilon) \right] \\ &< 0, \end{aligned}$$

which again is a contradiction. Thus,

$$D^+S_l(t) < 0$$

for $t \geq T_2$, and so

$$\lim_{t \rightarrow +\infty} S_l(t) = \sigma_l.$$

It follows from (2.1) for $t \geq T_2$ that

$$\begin{aligned} D^+ \left| \ln \frac{x_l(t) + x_l^*}{x_l^*} \right| &\leq |a_{ll}| |x_l(t)| + \sum_{j=1}^n [|a_{lj}| (1 - \delta_{lj}) |x_j(t)| \\ &\quad + |b_{lj}| \cdot |x_j(t - \tau_1(t))| + |c_{lj}| \cdot |\dot{x}_j(t - \tau_2(t))|] \\ &\leq -\alpha_l |a_{ll}| \left[S_l(t) - \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \cdot \omega_{lh} \cdot (\sigma_l + h) \right] \\ &\leq -\alpha_l |a_{ll}| \left[(\sigma_l - \varepsilon) - \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \omega_{lh} (\sigma_l + h) \right] \end{aligned}$$

and we see that

$$\begin{aligned} \left| \ln \frac{x_l(t) + x_l^*}{x_l^*} \right| &\leq \left| \ln \frac{x_l(T_2) + x_l^*}{x_l^*} \right| - (t - T_2) \alpha_l |a_{ll}| \\ &\quad \times \left[(\sigma_l - \varepsilon) - \frac{1}{\alpha_l} \sum_{h=1}^{2n} \alpha_h \omega_{lh} (\sigma_l + \varepsilon) \right] \\ &\rightarrow -\infty \end{aligned}$$

as $t \rightarrow +\infty$. This is a contradiction.

Thus, $\sigma_l = 0$ and so

$$|x_i(t)| \rightarrow 0, \quad |\dot{x}_i(t)| \rightarrow 0 \quad (i = 1, \dots, n)$$

as $t \rightarrow +\infty$. The proof is complete.

4. EXAMPLES

EXAMPLE 4.1. Consider the one-dimensional system

$$\dot{x}(t) = [x(t) + 1] \cdot [ax(t) + bx(t - \tau) + c\dot{x}(t - \tau)], \quad t \geq 0, \quad (4.1)$$

where $a < 0, b, c$ and $\tau > 0$ are constants. It is easy to check that

$$\left| \frac{b}{a} \right| + 2|c| < 1 \quad (4.2)$$

implies

$$\rho \begin{pmatrix} \frac{|b|}{|a|} & \frac{|c|}{|a|} \\ |a| + |b| & |c| \end{pmatrix} < 1.$$

Hence, from Theorem 3.1, the zero solution of (4.1) is uniformly asymptotically stable if (4.2) holds.

EXAMPLE 4.2. Consider the system

$$\begin{cases} \dot{x}_1(t) = [x_1(t) + 1] \cdot \left[-4x_1(t) + x_2\left(t - \frac{t}{2}\right) - \frac{1}{4}\dot{x}_1\left(t - \frac{t}{2}\right) \right] \\ \dot{x}_2(t) = [x_2(t) + 1] \left[\dot{x}_1\left(t - \frac{t}{2}\right) - 4x_2(t) \right], \end{cases} \quad t \geq 0. \quad (4.3)$$

We have

$$\Omega = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{16} & 0 \\ \frac{1}{4} & 0 & 0 & 0 \\ 4 & 1 & \frac{1}{4} & 0 \\ 1 & 4 & 0 & 0 \end{pmatrix}.$$

It is easy to calculate that $\rho(\Omega) < 1$. Then, by Theorem 3.2 the zero solution of (4.3) is asymptotically stable.

ACKNOWLEDGMENT

I thank the referees for careful reading of the paper and giving many useful suggestions.

REFERENCES

1. K. Gopalsamy and B. G. Zhang, On a neutral delay logistic equation, *Dynamics Stability Systems* **2** (1988), 183–195.
2. Y. Kuang, On neutral delay logistic Gause-type predator-prey systems, *Dynamics Stability Systems* **6** (1991), 173–189.
3. Y. Kuang, On neutral delay two-species Lotka-Volterra competitive systems, *J. Austral. Math. Soc. Ser. B* **32** (1991), 311–326.
4. Y. Kuang and A. Feldstein, Boundedness of solutions of a nonlinear nonautonomous neutral delay equation, *J. Math. Anal. Appl.* **156** (1991), 193–204.